Energy-momentum current for coframe gravity

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Abstract. The obstruction for the existence of an energy momentum tensor for the gravitational field is connected with differential-geometric features of the Riemannian manifold. It has not to be valid for alternative geometrical structures.
A teleparallel manifold is defined as a parallelizable differentiable $4D$-manifold endowed with a class of smooth coframe fields related by global Lorentz, i.e., $SO(1,3)$ transformations. In this article a general free parametric class of teleparallel models is considered. It includes a 1-parameter subclass of viable models with the Schwarzschild coframe solution.
A new form of the coframe field equation is derived from the general teleparallel Lagrangian by introducing the notion of a 3-parameter conjugate field strength $\mathcal{F}^a$. The field equation turns out to have a form completely similar to the Maxwell field equation $d*\mathcal{F}^a = \mathcal{T}^a$. By applying the Noether procedure, the source 3-form $\mathcal{T}^a$ is shown to be connected with the diffeomorphism invariance of the Lagrangian. Thus the source $\mathcal{T}^a$ of the coframe field is interpreted as the total conserved energy-momentum current. The energy-momentum tensor for coframe is defined. The total energy-momentum current of a system of a coframe and a material fields is conserved. Thus a redistribution of the energy-momentum current between a material and a coframe (gravity) fields is possible in principle, unlike as in the standard GR.
For special values of parameters, when the GR is reinstated, the energy-momentum tensor gives up the invariant sense, i.e., becomes a pseudo-tensor. Thus even a small-parametric change of GR turns it into a well defined Lagrangian theory.

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1. Introduction

The concept of an energy-momentum tensor for the gravitational field is, undoubtedly, the most puzzling issue in general relativity (GR). Such a tensor of non-geometric material fields acting in a fixed geometrical background is well defined. This quantity (denote it by $T^{\mu}{}_{\nu}$) obeys the following properties. It is

(i) *local* - i.e., constructed only from the fields taken at an arbitrary point on a manifold and from the derivatives of these fields taken at the same point,

(ii) *diffeomorphic covariant* - i.e., transforms as a tensor under diffeomorphisms of the manifold,

(iii) *inner invariant* - i.e., does not change under inner symmetry transformations of fields, which preserve the Lagrangian,

(iv) *conserved* - i.e., satisfies the covariant divergence equation $T^{\mu}{}_{\nu;\mu} = 0$,

(v) *"the first integral of the field equation"* - it is derivable from the field equations by integration. The order of field derivatives in $T^{\mu}{}_{\nu}$ is of one less than the order of the field equation.

It is well known that in GR an energy-momentum tensor of the metric (gravitational) field itself, satisfying the conditions listed above, fails to exist. This fact is usually related to the equivalence principle, which implies that the gravitational field can not be detected at a point as a covariant object. This conclusion can also be viewed as a purely differential-geometric fact. Indeed, the components of the metric tensor are managed by a system of second order partial differential equations. Thus the energy-momentum quantity has to be a local tensor constructed from the metric components and their first order derivatives. The corresponding theorem of (pseudo) Riemannian geometry states that every expression of such a type is trivial. Thus, the objection for the existence of a gravitational energy-momentum tensor is directly related to the geometric properties of the (pseudo) Riemannian manifold. It is natural to expect that this objection can be lifted in an alternative model of gravity, even connected with the geometry of the manifold.

In resent time *teleparallel structures* on spacetime have evoked a considerable interest for various reasons. They was considered as an essential part of generalized non-Riemannian theories such as the Poincaré gauge theory (see [1] - [5] and the references therein) or metric - affine gravity [6] as well as a possible physical relevant geometry by itself - teleparallel description of gravity (see Refs. [8] to [18]) . Another important subject are the various applications of the frame technique in physical theories based on classical (pseudo) Riemannian geometry. For instant in [19] teleparallel approach used for positive-gravitational-energy proof. In [20] the relation between spinor Lagrangian and teleparallel theory is established.

The most important property of the teleparallel theory is the existence of a family of viable gravitational models.

In the present paper we study the general free-parametric model on differential manifold, endowed with a metric constructed from the coframe. We start with a brief survey of the coframe teleparallel approach to gravity.

Our main results are presented in the third section. We consider a pure coframe field with the most general odd quadratic coframe Lagrangian, which involves 3 dimensionless parameters ρ_1, ρ_2, ρ_3. The field equation is derived in a form almost literally similar to the Maxwell-Yang-Mills field equation. The source term of the equation is a conserved vector valued 3-form.

By applying the Noether procedure, this 3-form is associated with the diffeomorphism invariance of the Lagrangian is derived. Hence, it is interpreted as the energy-momentum current of the coframe field. The notion of the Noether current and the Noether charge

for the coframe field are introduced.
The energy-momentum tensor is defined as a map of the module of current 3-forms into the module of vector fields. Thus, the energy-momentum tensor for the coframe field is defined in a diffeomorphism invariant and a translational covariant way.
For a system of a coframe and a material field it is shown that the total energy-momentum current serves as a source of the coframe field. This total current is conserved. Consequently, a redistribution of energy between material and gravitational (coframe) fields is possible in principle.
We briefly discuss the special case of the teleparallel equivalent of GR. This models turns to be an alternative formulation of GR, not an alternative model. The energy-momentum current in this case loses the invariant sense, with accordance with numerous investigations in standard GR.
Consequently, all the viable teleparallel models with $\rho_1 = 0$, except of the GR, have a well-defined energy-momentum tensor.

2. Teleparallel gravity

Let us give a brief account of gravity on teleparallel manifolds. Consider a coframe field $\{\vartheta^a,\ a = 0,1,2,3\}$ defined on a $4D$ differential manifold M. The 1-forms ϑ^a are declared to be pseudo-orthonormal. This determines completely a metric on the manifold M by

$$g = \eta_{ab}\vartheta^a \otimes \vartheta^b. \tag{2.1}$$

So, the coframe field ϑ^a is considered as a basic dynamical variable while the metric g is treated as only a secondary structure.
Such simple coframe structure is not complying with the relativistic paradigm because the coframe 1-forms ϑ^a produce peculiar directions at every point on M. In order to have an isotropic structure the coframe variable have to be defined only up to *global pseudo-rotations*, i.e. $SO(1,3)$ transformations. Consequently, the truly dynamical variable is the equivalence class of coframes $[\vartheta^a]$, while the global pseudo-rotations produce the equivalence relation on this class. Hence, in addition to the invariance under the diffeomorphic transformations of the manifold M, the basic geometric structure has to be global $SO(1,3)$ invariant.
The well known property of the teleparallel geometry is the possibility to define the parallelism of two vectors at different points by comparing the components of the vectors in local frames. Namely, two vectors (1-forms) are parallel if the corresponding components referred to a local frame (coframe) are proportional. This *absolute parallelism* structure produces a global path independent parallel transport. In the affine-connections formalism such a transport is described by existence of a special teleparallel connections of vanishing curvature [6]. However, the Riemannian curvature of the manifold, which is constructed from the metric (2.1), is non-zero, in general.
Gravity is described by the teleparallel geometry in a way similar to Einstein theory, i.e., by differential-geometric invariants of the structure. Looking for such invariants,

an important distinction between the metric and the teleparallel structures emerges.
The metric structure admits diffeomorphic invariants only of the second order or greater. The metric invariants of the first order are trivial. The unique invariant of the second order is the scalar curvature. This expression is well known to play the role of an integrand in the Einstein-Hilbert action.
The teleparallel structure admits diffeomorphic and $SO(1,3)$ global invariants even of the first order. A simple example is the expression $e_a\rfloor d\vartheta^a$. The diffeomorphic invariant and global covariant operators, which can contribute to a general field equation, constitute a rich class [16].
Restrict the consideration to odd, quadratic (in the first order derivatives of the coframe field ϑ^a), diffeomorphic, and global $SO(1,3)$ invariant Lagrangians. A general Lagrangian of such type is represented by a linear combination of three Weitzenböck quadratic teleparallel invariants. The symmetric form of this Lagrangian is [13] ($\ell =$ Planck length)

$$L^{cof}=\frac{1}{2\ell^2}\sum_{i=1}^{3}\rho_i\,{}^{(i)}L, \tag{2.2}$$

with

$$^{(1)}L=d\vartheta^a\wedge *d\vartheta_a, \tag{2.3}$$

$$^{(2)}L=\Big(d\vartheta_a\wedge\vartheta^a\Big)\wedge *\Big(d\vartheta_b\wedge\vartheta^b\Big), \tag{2.4}$$

$$^{(3)}L=(d\vartheta_a\wedge\vartheta^b)\wedge *\Big(d\vartheta_b\wedge\vartheta^a\Big). \tag{2.5}$$

The 1-forms ϑ^a are assumed to carry the dimension of length, while the coefficients ρ_i are dimensionless. Hence the total Lagrangian L^{cof} is dimensionless. In order to simplify the formulas below we will use the Lagrangian $L=\ell^2L^{cof}$ of the dimension: length square. In other worlds the geometrized units system $G=c=\hbar=1$ is applied. For comparison with the ordinary units see [21].
Every term of the Lagrangian (2.2) is independent of a specific choice of a coordinate system and invariant under a global (rigid) $SO(1,3)$ transformation of the coframe. Thus, different choices of the free parameters ρ_i yield different translational and diffeomorphic invariant classical field models. Some of them are known to be applicable for description of gravity.
The field equation is derived from the Lagrangian (2.2) in the form [9],[13]

$$\begin{aligned}&\rho_1\Big(2d*d\vartheta_a+e_a\rfloor(d\vartheta^b\wedge *d\vartheta_b)-2(e_a\rfloor d\vartheta^b)\wedge *d\vartheta_b\Big)+\\&\rho_2\Big(-2\vartheta_a\wedge d*(d\vartheta^b\wedge\vartheta_b)+2d\vartheta_a\wedge *(d\vartheta^b\wedge\vartheta_b)+\\&e_a\rfloor\Big(d\vartheta^c\wedge\vartheta_c\wedge *(d\vartheta^b\wedge\vartheta_b)\Big)-2(e_a\rfloor d\vartheta^b)\wedge\vartheta_b\wedge *(d\vartheta^c\wedge\vartheta_c)\Big)+\\&\rho_3\Big(-2\vartheta_b\wedge d*(\vartheta_a\wedge d\vartheta^b)+2d\vartheta_b\wedge *(\vartheta_a\wedge d\vartheta^b)+\\&e_a\rfloor\Big(\vartheta_c\wedge d\vartheta^b\wedge *(d\vartheta^c\wedge\vartheta_b)\Big)-2(e_a\rfloor d\vartheta^b)\wedge\vartheta_c\wedge *(d\vartheta^c\wedge\vartheta_b)\Big)=0.\end{aligned} \tag{2.6}$$

The general ("diagonal") spherical-symmetric static solution to the field equation (2.6) for all possible values of ρ_i is derived [17]. It turns out that $\rho_1 = 0$ is a necessary and sufficient condition to have a solution with Newtonian behavior at infinity. The coframe solution in this case is unique and yields via (2.1) the Schwarzschild metric. In such a way by rejecting the pure Yang-Mills-type term (2.3) the model turns out to be a viable model for gravity.

Few remarks on the analytic structure of the field equation (2.6) are now in order.

On one hand, the coframe field is a complex of 16 independent variables while the symmetric metric tensor field has only 10 independent components. The remaining 6 components are related to the spinorial properties of the field. An additional *local* $SO(1,3)$ *invariance*, which appears in the case

$$\rho_1 = 0, \qquad \rho_2 + 2\rho_3 = 0, \tag{2.7}$$

restricts the set of 16 independent variables to a subset of 10 variables. This subset is in one to one correspondence with 10 independent components of the metric.

On the other hand the field equation (2.6) is a system of 16 independent equations. This system is reduced to two covariant systems - a symmetric tensorial sub-system of 10 independent equations and an antisymmetric tensorial sub-system of 6 independent equations. In the case (2.6) (and only in this case) the antisymmetric equation vanishes identically and the system is restricted to a system of 10 independent equations for 10 independent variables. Therefore the local $SO(1,3)$ invariant coframe structure coincides with the metric structure. The model with parameters (2.7) is referred to as the *teleparallel equivalent of gravity.* This local invariant construction, in fact, is not an alternative model of gravity but merely an alternative coframe reformulation of the standard (tensorial) GR. Namely, the Lagrangian (2.2) with the parameters determined by (2.7) coincides with the Hilbert-Einstein Lagrangian (up to total derivative terms) [7].

In the general case, when the relations (2.7) do not hold, the field equation (2.6) is a well defined covariant system of 16 independent equations for 16 independent variables. Certainly the most interesting case is

$$\rho_1 = 0, \qquad \rho_2, \rho_3 \quad \text{- } \textit{arbitrary} \tag{2.8}$$

For these values of parameters the Lagrangian (2.2) and, consequently, the field equations (2.6) describe a 1-parametric family of models with a unique "diagonal" spherical-symmetric solution which yields the Schwarzschild metric. Hence all the models of the family conform to the observation data at least for the three classical tests of GR. Thus the family of models (2.8) provides a viable alternative to the standard GR.

3. Coframe field

3.1. The compact form of the Lagrangian

We study an even smooth coframe field ϑ^a defined on a differential $4D$-manifold M. Our goal is to derive a conserved current expression for this coframe field in a set of models parameterized by the constants ρ_i. Although there are good physical reasons for rejecting the pure Yang-Mills term in the Lagrangian by taking $\rho_1 = 0$, the general case is not more difficult for treatment, so we will consider the complete set of teleparallel models (2.2) with arbitrary values of parameters.

The standard computations of the variation of a Lagrangian defined on a teleparallel manifold are rather complicated [23], [13]. It is because one needs to vary not only the coframe ϑ^a itself, but also the the dual frame e_a and even the Hodge dual operator $*$, that depends on the pseudo-orthonormal coframe implicitly.

In order to avoid these technical problems we will rewrite the total Lagrangian (2.2) in a compact form which will be useful for the variation procedure.

Consider the exterior differentials of the basis 1-forms $d\vartheta^a$ and introduce the C-coefficients of their expansion in the basis of even 2-forms ϑ^{ab} (here and later the abbreviation $\vartheta^{ab\cdots} = \vartheta^a \wedge \vartheta^b \wedge \cdots$ is used)

$$d\vartheta^a = \vartheta^a_{\beta,\alpha} dx^\alpha \wedge dx^\beta := \frac{1}{2} C^a{}_{bc} \vartheta^{bc}. \tag{3.1}$$

By definition, the coefficients $C^a{}_{bc}$ are antisymmetric: $C^a{}_{bc} = -C^a{}_{cb}$. Their explicit expression is derived straightforward from the definition (3.1)

$$C^a{}_{bc} := e_c \rfloor (e_b \rfloor d\vartheta^a). \tag{3.2}$$

In terms of the C-coefficients the independent parts of the Lagrangian (2.2) are

$$\begin{aligned}
{}^{(1)}L &= \frac{1}{2} C_{abc} C^{abc} * 1, \\
{}^{(2)}L &= \frac{1}{2} C_{abc} \Big(C^{abc} + C^{bca} + C^{cab} \Big) * 1, \\
{}^{(3)}L &= \frac{1}{2} \Big(C_{abc} C^{abc} - 2 C^a{}_{ac} C_b{}^{bc} \Big) * 1.
\end{aligned} \tag{3.3}$$

Note that the form (3.3) is useful for a proof of the completeness of the set of quadratic invariants [16]. It is enough to consider all the possible combinations of the indices. Thus a linear combination of the Lagrangians (3.3) is the most general quadratic coframe Lagrangian.

Using (3.3) we rewrite the coframe Lagrangian in a compact form

$$L = \frac{1}{4} C_{abc} C_{ijk} \lambda^{abcijk} * 1, \tag{3.4}$$

where the constant symbols

$$\begin{aligned}
\lambda^{abcijk} := {} & (\rho_1 + \rho_2 + \rho_3) \eta^{ai} \eta^{bj} \eta^{ck} + \rho_2 (\eta^{aj} \eta^{bk} \eta^{ci} + \eta^{ak} \eta^{bi} \eta^{cj}) \\
& - 2\rho_3 \eta^{ac} \eta^{ik} \eta^{bj}
\end{aligned} \tag{3.5}$$

are introduced. It can be checked, by straightforward calculation, that these λ-symbols are invariant under a transposition of the triplets of indices:

$$\lambda^{abcijk} = \lambda^{ijkabc}. \tag{3.6}$$

We also introduce an abbreviated notation

$$F^{abc} := \lambda^{abcijk} C_{ijk}. \tag{3.7}$$

The total Lagrangian (2.2) reads now as

$$L = \frac{1}{4} C_{abc} F^{abc} * 1. \tag{3.8}$$

This form of the Lagrangian will be used in the consequence for the variation procedure. The Lagrangian (3.8) can also be rewritten in a component free notations.
Define one-indexed 2-forms: a strength form

$$\mathcal{C}^a := \frac{1}{2} C^{abc} \vartheta_{bc} = d\vartheta^a. \tag{3.9}$$

and a conjugate strength form $\mathcal{F}^a := \frac{1}{2} F^{abc} \vartheta_{bc}$

$$\mathcal{F}^a = (\rho_1 + \rho_3)\mathcal{C}^a + \rho_2 e^a \rfloor (\vartheta^m \wedge \mathcal{C}_m) - \rho_3 \vartheta^a \wedge (e_m \rfloor \mathcal{C}^m) \tag{3.10}$$

The 2-form $\mathcal{F}^a$ can be also represented via the irreducible (under the Lorentz group) decomposition of the 2-form $\mathcal{C}^a$ (see [13], [24]). Write

$$\mathcal{C}^a = {}^{(1)}\mathcal{C}^a + {}^{(2)}\mathcal{C}^a + {}^{(3)}\mathcal{C}^a, \tag{3.11}$$

where

$$\begin{aligned} {}^{(1)}\mathcal{C}^a &= \mathcal{C}^a - {}^{(2)}\mathcal{C}^a - {}^{(3)}\mathcal{C}^a, \\ {}^{(2)}\mathcal{C}^a &= \frac{1}{3} \vartheta^a \wedge (e_m \rfloor \mathcal{C}^m), \\ {}^{(3)}\mathcal{C}^a &= \frac{1}{3} e^a \rfloor (\vartheta_m \wedge \mathcal{C}^m). \end{aligned} \tag{3.12}$$

Substitute (3.12) into (3.10) to obtain

$$\mathcal{F}^a = (\rho_1 + \rho_3)^{(1)}\mathcal{C}^a + (\rho_1 - 2\rho_3)^{(2)}\mathcal{C}^a + (\rho_1 + 3\rho_2 + \rho_3)^{(3)}\mathcal{C}^a. \tag{3.13}$$

The coefficients in (3.13) coincide with those calculated in [13].
The 2-forms $\mathcal{C}^a$ and $\mathcal{F}^a$ do not depend on a choice of a coordinate system. They change as vectors by global $SO(1,3)$ transformations of the coframe. Using (3.9) the coframe Lagrangian can be rewritten as

$$L = \frac{1}{2} \mathcal{C}_a \wedge *\mathcal{F}^a \tag{3.14}$$

Observe that the Lagrangian (3.14) is of the same form as the standard electromagnetic Lagrangian $L = \frac{1}{2} F \wedge *F$. However, the teleparallel Lagrangian involves the vector valued 2-forms of the field strength, while the electromagnetic Lagrangian is constructed of the the scalar valued 2-forms.

3.2. Variation of the Lagrangian

The Lagrangian (3.14) depends on the coframe field ϑ^a and on its first order derivatives only. Thus the first order variation formalism guarantee the corresponding Euler-Lagrange equation to be of at most second order. Consider the variation of the coframe Lagrangian, taken in the component-wise form (3.8), relative to small independent variations of the 1-forms ϑ^a. The λ-symbols (3.5) are constants and obey the symmetry property (3.6). Thus

$$C_{abc}\delta F^{abc} = C_{abc}\lambda^{abcijk}\delta C_{ijk} = \delta C_{abc}F^{abc} \tag{3.15}$$

Consequently the variation of the Lagrangian (3.8) takes the form

$$\delta L = \frac{1}{2}\delta C_{abc}F^{abc} * 1 - L * \delta(*1). \tag{3.16}$$

The variation of the volume element is

$$\begin{aligned}\delta(*1) = &- \delta(\vartheta^{0123}) = -\delta\vartheta^0 \wedge \vartheta^{123} - \cdots = -\delta\vartheta^0 \wedge *\vartheta^0 - \cdots \\ &= \delta\vartheta^m \wedge *\vartheta_m.\end{aligned}$$

Thus

$$L * \delta(*1) = (\delta\vartheta^m \wedge *\vartheta_m) * L = -\delta\vartheta^m \wedge (e_m \rfloor L). \tag{3.17}$$

As for the variation of the C-coefficients, we calculate them by equating the variations of the two sides of the equation (3.1)

$$\delta d\vartheta_a = \frac{1}{2}\delta C_{amn}\vartheta^{mn} + C_{amn}\delta\vartheta^m \wedge \vartheta^n. \tag{3.18}$$

Use the formulas (A.12) and (A.15) to derive

$$\begin{aligned}\delta d\vartheta_a \wedge *\vartheta_{bc} = &\frac{1}{2}\delta C_{amn}\vartheta^{mn} \wedge *\vartheta_{bc} + C_{amn}\delta\vartheta^m \wedge \vartheta^n \wedge *\vartheta_{bc} \\ &= -\frac{1}{2}\delta C_{amn}\vartheta^m \wedge *(e^n \rfloor \vartheta_{bc}) - C_{amn}\delta\vartheta^m \wedge *(e^n \rfloor \vartheta_{bc}) \\ &= \delta C_{abc} * 1 - 2\delta\vartheta^m \wedge C_{am[b} * \vartheta_{c]}.\end{aligned}$$

Therefore

$$\delta C_{abc} * 1 = \delta(d\vartheta_a) \wedge *\vartheta_{bc} + 2\delta\vartheta^m \wedge C_{am[b} * \vartheta_{c]}. \tag{3.19}$$

After substituting (3.17–3.19) into (3.16) the variation of the Lagrangian takes the form

$$\delta L = \frac{1}{2}F^{abc}\Big(\delta(d\vartheta_a) \wedge *\vartheta_{bc} + 2\delta\vartheta^m \wedge C_{am[b} * \vartheta_{c]}\Big) + \delta\vartheta^m \wedge (e_m \rfloor L).$$

Extract total derivatives to obtain

$$\begin{aligned}\delta L = &\frac{1}{2}\delta\vartheta_m \wedge \Big(d(*F^{mbc}\vartheta_{bc}) + 2F^{abc}C_{am[b} * \vartheta_{c]} + 2e_m \rfloor L\Big) \\ &+ \frac{1}{2}d\Big(\delta\vartheta_a \wedge *F^{abc}\vartheta_{bc}\Big).\end{aligned} \tag{3.20}$$

The variation relation (3.20) will play a basic role in the sequel. Let us rewrite it in a compact form by using the 2-forms (3.9) and (3.10). The terms of the form $F \cdot C$ can be rewritten as

$$\begin{aligned} F^{abc}C_{am[b} * \vartheta_{c]} &= (F^{abc} - F^{acb})C_{am[b} * \vartheta_{c]} \\ &= C_{amb} * (e^b \rfloor \mathcal{F}^a) = -(e_m \rfloor \mathcal{C}_a) \wedge *\mathcal{F}^a. \end{aligned}$$

Hence, (3.20) takes the form

$$\delta L = \delta\vartheta^m \wedge \Big(d(*\mathcal{F}_m) - (e_m \rfloor \mathcal{C}_a) \wedge *\mathcal{F}^a + e_m \rfloor L\Big) + d(\delta\vartheta^m \wedge \mathcal{F}_m). \quad (3.21)$$

Collect now the quadratic terms into a differential 3-form

$$\mathcal{T}_m := (e_m \rfloor \mathcal{C}_a) \wedge *\mathcal{F}^a - e_m \rfloor L. \quad (3.22)$$

Consequently, the variational relation (3.20) results in the final form

$$\delta L = \delta\vartheta^m \wedge \Big(d * \mathcal{F}_m - \mathcal{T}_m\Big) + d(\delta\vartheta^m \wedge \mathcal{F}_m). \quad (3.23)$$

3.3. The field equations

We are ready now to write down the field equations. Consider independent free variations of a coframe field vanishing at infinity (or at the boundary of the manifold ∂M). The variational relation (3.23) yields *the coframe field equation*

$$d * \mathcal{F}^m = \mathcal{T}^m. \quad (3.24)$$

Note that this is the same equation as (2.6) because it was obtained from the same Lagrangian by the same free variations of the coframe. The equivalence of the forms is shown in the Appendix B.

Observe that the structure of coframe field equation is formally similar to the structure of the standard electromagnetic field equation $d * F = J$. Namely, the left hand side of both equations is the exterior derivative of the dual strength field while the right hand side is an odd 3-form. Thus the 3-forms $\mathcal{T}^m$ serves as a source for the strength field $\mathcal{F}^m$, as well as the 3-form of electromagnetic current J is a source for the electromagnetic field.

There are, however, important distinctions:

i) The coframe field current $\mathcal{T}_m$ is a vector-valued 3-form while the electromagnetic current J is scalar-valued.

ii) The field equation (3.24) is nonlinear.

iii) The electromagnetic current J depends on an exterior material field, while the coframe current $\mathcal{T}^m$ is interior (depends on the coframe itself).

The exterior derivation of the field equation (3.24) yields the conservation law

$$d\mathcal{T}_m = 0. \quad (3.25)$$

Note, that this equation obeys all symmetries of the Lagrangian. It is diffeomorphism invariant and global $SO(1,3)$ covariant. Thus we obtain a conserved total 3-form (3.22) which is constructed from the first order derivatives of the field variables (coframe). It is local and covariant. The 3-form $\mathcal{T}_m$ is our candidate for the coframe energy-momentum current.

3.4. Conserved currents

The current $\mathcal{T}_m$ is obtained directly, i.e., by separation of the terms in the field equation. In order to identify the nature of this conserved 3-form we have to answer the question: *What symmetry this conserved current can be associated with?*
Return to the variational relation (3.23). On shell, for the fields satisfying the field equations (3.24), it takes the form

$$\delta L = d(\delta\vartheta^a \wedge *\mathcal{F}_a). \tag{3.26}$$

Consider the variations of the coframe field produced by the Lie derivative taken relative to a smooth vector field X, i.e.,

$$\delta\vartheta^a = \mathcal{L}_X\vartheta^a = d(X\rfloor\vartheta^a) + X\rfloor d\vartheta^a. \tag{3.27}$$

The Lagrangian (3.8) is a diffeomorphic invariant, hence it's variation is produced by the Lie derivative taken relative to the same vector field X, i.e.,

$$\delta L = \mathcal{L}_X L = d(X\rfloor L). \tag{3.28}$$

Thus the relation (3.26) takes a form of a conservation law $d\Theta(X)$ for the Nether 3-form

$$\Theta(X) := \Big(d(X\rfloor\vartheta^a) + X\rfloor\mathcal{C}^a\Big) \wedge *\mathcal{F}_a - X\rfloor L. \tag{3.29}$$

This quantity includes the derivatives of an arbitrary vector field X. Such a non-algebraic dependence of the conserved current is an obstacle for definition of an energy-momentum tensor. This problem is solved merely by using the canonical form of the current. Let us take $X = e_a$. The first term of (3.29) vanishes identically. Thus

$$\Theta(e_m) = (e_m\rfloor\mathcal{C}^a) \wedge *\mathcal{F}_a - e_m\rfloor L. \tag{3.30}$$

Observe that the right hand side of the equation (3.30) is exactly the same expression as the source term of the field equation (3.24):

$$\Theta(e_m) = \mathcal{T}_m \tag{3.31}$$

Thus the conserved current $\mathcal{T}_m$ defined in (3.22) is associated with the diffeomorphism invariance of the Lagrangian. Consequently the vector-valued 3-form (3.22) represents the energy-momentum current of the coframe field.

3.5. Noether charge

Let us look for an additional information incorporated in the conserved current (3.29). Extract the total derivative to obtain

$$\Theta(X) = d\Big((X\rfloor\vartheta^a) * \mathcal{F}_a\Big) - (X\rfloor\vartheta^a)(d * \mathcal{F}_a - \mathcal{T}_a) \tag{3.32}$$

Thus, up to the field equation (3.24), the current $\mathcal{T}(X)$ represents a total derivative of a certain 2-form $\Theta(X) = dQ(X)$. This result is a special case of a general proposition due to Wald [22] for a diffeomorphic invariant Lagrangians. The 2-form

$$Q(X) = (X\rfloor\vartheta^a) * \mathcal{F}_a. \tag{3.33}$$

is referred to as the *Noether charge for the coframe field.* Consider $X = e_a$ and denote $Q_a := Q(e_a)$. From (3.33) we obtain that this canonical Noether charge of the coframe field coincides with the dual of the conjugate strength.

$$Q_a = Q(e_a) = *\mathcal{F}_a. \tag{3.34}$$

In this way the 2-form $\mathcal{F}_a$, which was used above only as a technical device for expressing the equations in a compact form, obtained now a meaningful description. Note, that the Noether charge plays an important role in Wald's treatment of the black hole entropy [22].

3.6. Energy-momentum tensor

In this section we construct an expressions for the energy-momentum tensor for the coframe field. Let us first introduce the notion of the energy-momentum tensor by the differential-form formalism. We are looking for a second rank tensor field of a type $(0, 2)$. Such a tensor can always be treated as a bilinear map $T: \ \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{F}(M)$, where $\mathcal{F}(M)$ is the algebra of C^∞-functions on M while $\mathcal{X}(M)$ is the $\mathcal{F}(M)$-module of vector fields on M. The unique way to construct a scalar from a 3-form and a vector is is to take the Hodge dual of the 3-form and to contract the result by the vector. Consequently, we define the energy-momentum tensor as

$$T(X,Y) := Y\rfloor * \mathcal{T}(X). \tag{3.35}$$

Observe that this quantity is a tensor if and only if the 3-form current $\mathcal{T}$ depends linearly (algebraic) on the vector field X. Certainly, $T(X,Y)$ is not symmetric in general. The antisymmetric part of the energy-momentum tensor is known from the Poincaré gauge theory [2] to represent the spinorial current of the field.
The canonical form of the energy-momentum $T_{ab} := T(e_a, e_b)$ tensor is

$$T_{ab} = e_b\rfloor * \mathcal{T}_a. \tag{3.36}$$

Another useful form of this tensor can be obtained from (3.36) by applying the rule (A.15)

$$T_{ab} = *(\mathcal{T}_a \wedge \vartheta_b). \tag{3.37}$$

The familiar procedure of rising the indices by the Lorentz metric η^{ab} produces two tensors of a type $(1, 1)$

$$T_a{}^b = *(\mathcal{T}_a \wedge \vartheta^b), \quad \text{and} \quad T^a{}_b = *(\mathcal{T}^a \wedge \vartheta_b), \tag{3.38}$$

which are different, in general. By applying the rule (A.9) the first relation of (3.38) is converted into

$$\mathcal{T}_a = T_a{}^b * \vartheta_b. \tag{3.39}$$

Thus, the components of the energy-momentum tensor are regarded as the coefficients of the current $\mathcal{T}_a$ in the dual basis $*\vartheta^a$ of the vector space Ω^3 of odd 3-forms.
In order to show that (3.39) conforms with the intuitive notion of the energy-momentum

tensor let us represent on the flat manifold the 3-form conservation law as a tensorial conservation law. Take a closed coframe $d\vartheta^a = 0$, thus $d*\vartheta_b = 0$. From (3.39) we derive

$$d\mathcal{T}_a = dT_a{}^b \wedge *\vartheta_b = -T_a{}^b{}_{,b} * 1.$$

Hence the differential-form conservation law $d\mathcal{T}_a = 0$ is equivalent to the tensorial conservation law $T_a{}^b{}_{,b} = 0$.

Apply now the definition (3.36) to the conserved current (3.22) for the coframe field. The energy-momentum tensor $T_{mn} = e_n \rfloor * \mathcal{T}_m$ is derived in the form

$$T_{mn} = e_n \rfloor * \Big((e_m \rfloor \mathcal{C}_a) \wedge *\mathcal{F}^a - \frac{1}{2} e_m \rfloor (\mathcal{C}_a \wedge *\mathcal{F}^a) \Big). \tag{3.40}$$

Using (A.15) we rewrite the first term in (3.40) as

$$e_n \rfloor * \Big((e_m \rfloor \mathcal{C}_a) \wedge *\mathcal{F}^a \Big) = - * \Big((e_m \rfloor \mathcal{C}_a) \wedge *(e_n \rfloor \mathcal{F}^a) \Big).$$

As for the second term in (3.40) it takes the form

$$-\frac{1}{2} e_n \rfloor * \Big(e_m \rfloor (\mathcal{C}_a \wedge *\mathcal{F}^a) \Big) = \frac{1}{2} \eta_{mn} * (\mathcal{C}_a \wedge *\mathcal{F}^a).$$

Consequently the energy-momentum tensor for the coframe field is

$$T_{mn} = - * \Big((e_m \rfloor \mathcal{C}_a) \wedge *(e_n \rfloor \mathcal{F}^a) \Big) + \frac{1}{2} \eta_{mn} * (\mathcal{C}_a \wedge *\mathcal{F}^a). \tag{3.41}$$

Observe that this expression is formally similar to the familiar expression for the energy momentum tensor of the Maxwell electromagnetic field:

$$T_{mn} = - * \Big((e_m \rfloor F) \wedge *(e_n \rfloor F) \Big) + \frac{1}{2} \eta_{mn} * (F \wedge *F). \tag{3.42}$$

The form (3.42) is no more than an expression of the electromagnetic energy-momentum tensor in arbitrary frame. In a specific coordinate chart $\{x^\mu\}$ it is enough to take the coordinate basis vectors $e_a = \partial_\alpha$ and consider $T_{\alpha\beta} := {}^{(e)}T(\partial_\alpha, \partial_\beta)$ to obtain the familiar expression

$$T_{\alpha\beta} = -F_{\alpha\mu} F_\beta{}^\mu + \frac{1}{4} \eta_{\alpha\beta} F_{\mu\nu} F^{\mu\nu}. \tag{3.43}$$

The electromagnetic energy-momentum tensor is obviously traceless. The same property holds also for the coframe field tensor.

Proposition *For all teleparallel models described by the Lagrangian (2.2), i.e., for all values of the parameters ρ_i, the energy-momentum tensor defined by (3.41) is traceless.*

Proof Compute the trace $T^m{}_m = T_{mn}\eta^{mn}$ of (3.41):

$$\begin{aligned} T^m{}_m &= - * \Big((e_m \rfloor \mathcal{C}_a) \wedge *(e^m \rfloor \mathcal{F}^a) \Big) + 2 * (\mathcal{C}_a \wedge *\mathcal{F}^a) \\ &= * \Big((e_m \rfloor \mathcal{C}_a) \wedge *^2 (\vartheta^m \wedge *\mathcal{F}^a) \Big) + 2 * (\mathcal{C}_a \wedge *\mathcal{F}^a) \\ &= - * \Big(\vartheta^m \wedge (e_m \rfloor \mathcal{C}_a) \wedge *\mathcal{F}^a \Big) + 2 * (\mathcal{C}_a \wedge *\mathcal{F}^a) = 0 \end{aligned}$$

In the latter equality the relation (A.9) was used.

It is well known that the traceless of the energy-momentum tensor is associated with the scale invariance of the Lagrangian. The rigid (λ is a constant) scale transformation

$x^i \to \lambda x^i$, is considered acting on a material field as $\phi \to \lambda^d \phi$, where d is the dimension of the field. The transformation does not act, however, on the components of the metric tensor and on the frame (coframe) components. It is convenient to shift the change on the metric and on the frame (coframe) components: $g_{\mu\nu} \to \lambda^2 g_{\mu\nu}$, $\vartheta^a{}_\mu \to \lambda \vartheta^a{}_\mu$, and $e_a{}^\mu \to \lambda^{-1} \vartheta_a{}^\mu$ with no change of coordinates. In the coordinate free approach the difference between two approaches is neglected and the transformation is

$$g \to \lambda^2 g, \qquad \vartheta^a \to \lambda \vartheta^a, \qquad \text{and} \qquad e_a \to \lambda^{-1} e_a \tag{3.44}$$

The transformation law of the teleparallel Lagrangian is simple to obtain from the component-wise form (3.3). Under the transformation (3.44) the volume element changes as $*1 \to \lambda^4 * 1$. As for the C-coefficients, they transform due to (3.2) as $C^a{}_{bc} \to \lambda^{-1} C^a{}_{bc}$. Consequently, by (2.3), the transformation law of the Lagrangian 4-form is $L \to \lambda^2 L$, which is the same as for the Hilbert-Einstein Lagrangian $L_{HE} = R\sqrt{-g} d^4x \to \lambda^2 L_{HE}$. After rescaling the Planck length the scale invariance is reinstated. Hence, for the pure teleparallel model the energy-momentum tensor have to be traceless in accordance with the proposition above.

3.7. The field equation for a general system

The coframe field equation have been derived for a pure coframe field. Consider now a general minimally coupled system of a coframe field ϑ^a and a material field ψ. The material field can be a differential form of an arbitrary degree and can carry arbitrary number of exterior and interior indices. Take the total Lagrangian of the system to be of the form ($\ell =$ Planck length)

$$L = \frac{1}{\ell^2} L^{cof}(\vartheta^a, d\vartheta^a) + L^{mat}(\vartheta^a, \psi, d\psi), \tag{3.45}$$

where the coframe Lagrangian L^{cof}, defined by (2.2), is of dimension length square, while the material Lagrangian L^{mat} is dimensionless.

The minimal coupling means here the absence of coframe derivatives in the material Lagrangian. Take the variation of (3.45) relative to the coframe field ϑ^a to obtain

$$\delta L = \frac{1}{\ell^2} \delta\vartheta^a \wedge \Big(d * \mathcal{F}_a - \underset{a}{\mathcal{T}}^{cof} - \ell^2 \underset{a}{\mathcal{T}}^{mat} \Big), \tag{3.46}$$

where the 3-form of coframe current is defined by (3.24). The 3-form of material current is defined via the variation derivative of the material Lagrangian taken relative to the coframe field ϑ^a:

$$\underset{a}{\mathcal{T}}^{mat} := -\frac{\delta}{\delta\vartheta^a} L^{mat}. \tag{3.47}$$

Introduce the total current of the system $\underset{a}{\mathcal{T}}^{tot} = \underset{a}{\mathcal{T}}^{cof} + \ell^2 \underset{a}{\mathcal{T}}^{mat}$, which is of dimension length (mass). Consequently, the field equation for the general system (3.45) takes the form

$$d * \mathcal{F}_a = \underset{a}{\mathcal{T}}^{tot}. \tag{3.48}$$

Using the energy-momentum tensor (3.39) this equation can be rewritten in a tensorial form

$$e_b \rfloor * d * \mathcal{F}_a = T^{tot}_{ab}, \tag{3.49}$$

or equivalently

$$\vartheta_b \wedge d * \mathcal{F}_a = T^{tot}_{ab} * 1. \tag{3.50}$$

The conservation law for the total current $d\mathcal{T}_a = 0$ is a straightforward consequence of the field equation (3.48).
The form (3.48) of the field equation looks like the Maxwell field equation for the electromagnetic field $d * F = J$. Observe, however, an important difference.
The source term in the right hand side of the electromagnetic field equation depends only on external fields. In the absence of the external sources $J = 0$, the electromagnetic strength $*F$ is a closed form. As a consequence, its cohomology class interpreted as a charge of the source. The electromagnetic field itself is uncharged.
As for the coframe field strength $\mathcal{F}^a$ its source depends on the coframe and of its first order derivatives. Consequently, the 2-form $*\mathcal{F}^a$ is not closed even in absence of the external sources. Hence the gravitational field is massive (charged) itself.
On the other hand the tensorial form (3.49) of the teleparallel field equation is similar to the Einstein field equation for the metric tensor $G_{ab} = 8\pi T^{mat}_{ab}$. Indeed, the left hand side in both equations are pure geometric quantities. Again, the source terms in the field equations are different. The source of the Einstein gravity is the energy-momentum tensor only of the materials fields. The conservation of this tensor is a consequence of the field equation. Thus even if some meaningful conserved energy-momentum current for the metric field existed it would have been conserved regardless of the material field current. Consequently, any redistribution of the energy-momentum current between the material and gravitational fields is forbidden in the framework of the traditional Einstein gravity.
As for the coframe field equation, the total energy-momentum current plays a role of the source of the field. Consequently the coframe field is completely "self-interacted" - the energy-momentum current of the coframe field produces an additional field. The conserved current of the coframe-material system is the total energy-momentum current, not only the material current. Thus in the framework of general teleparallel construction the redistribution of the current between the material field and the coframe field is, in principle, possible.

4. Teleparallel equivalent of GR

The gravitational energy-momentum problem attracted recently a considerable interest in the framework of the teleparallel equivalent of GR model (denote it by $\mathrm{GR}_{||}$) []. As it was mentioned, this model corresponds to a special choice (2.7) of free parameters of the general teleparallel model described above. Let us start with a comparison between the differential form approach and the tensorial approach used in the $\mathrm{GR}_{||}$.

1) The basic dynamical variable of the $\mathrm{GR}_{||}$ is the frame (tetrad) field $h^a{}_\mu$, where the Greek index is related to the coordinates while the Latin index denotes the corresponding vector in the frame. Due to the canonical duality between 1-forms and vectors this set of variables is equivalent to the components of the coframe field taken in coordinate basis $\vartheta^a = \vartheta^a{}_\mu dx^\mu$.
2) The gravitational field strength of the $\mathrm{GR}_{||}$ is the torsion tensor $T^\rho{}_{\mu\nu} = h_a{}^\rho \partial_{[\nu} h^a{}_{\mu]}$. This object is in one to one correspondence with the coefficients of the 2-form $\mathcal{C}^a = d\vartheta^a$ taken in a coordinate basis. The second field strength tensor of the $\mathrm{GR}_{||}$, the contorsion tensor, is defined as a linear combination of the torsion tensor with the coefficients depend on the metric tensor $g^{\mu\nu}$. It corresponds to the linear combinations of the components $C^a{}_{bc}$, used above.
3) The Lagrangian of the $\mathrm{GR}_{||}$, its field equations and conserved current are constructed from the torsion and contorsion tensors and reinstated from the formulas above in a special case of parameters (2.7) being written in a coordinate basis.
Thus the two techniques: the tensorial representation of the $\mathrm{GR}_{||}$ and the differential form approach used above are principally equivalent. It should be noted, however, that the very fact of treating the gravitational strength as the antisymmetric tensor of torsion shows that the differential forms approach is a more appropriative mathematical device here.
Observe now the principal features of the $\mathrm{GR}_{||}$ model. Certainly in the framework of general coframe model the construction do not depends on the specific values of the parameters ρ_i. The $\mathrm{GR}_{||}$ Lagrangian is reinstated from the general Lagrangian (2.2) merely by inserting the specific values of the coefficients. Also the field equation and the conserved current do not depend on a choice of the parameters. Thus, it seems that the $\mathrm{GR}_{||}$-model can be considered as a simple limit $\rho_1 \to 0,\ \rho_2 + 2\rho_3 \to 0$ of the general teleparallel construction. A more detailed analyses shows, however, that it is not a case. Even the limit $\rho_1 \to 0$ is not trivial. Indeed, the typical form of the spherical-symmetric solution in the general model [17] is $\vartheta^a = (r/r_0)^\alpha dx^a$, where α depends of the coefficients ρ_i. The Schwarzschild coframe is a special solution, which appears in the case $\rho_1 = 0$ only. Certainly, the typical solution does not approach the Schwarzschild coframe in the limit. In fact the general free parametric model has a non-continuous dependence on the parameter ρ_1.
The second limit $\rho_2 + 2\rho_3 \to 0$ produces an additional degeneration of the general coframe construction. The Lagrangian (2.2) obtains in this case a higher symmetry. This is a local Lorentz invariance of the coframe field. Due to the known theorem of the variational calculus this invariance appears also on the field equation level. However, the separation of the field equation to the total derivative term $d*\mathcal{F}^a$ and the conserved current term $\mathcal{T}^a$ is not local Lorentz invariant. As a result in the $\mathrm{GR}_{||}$ limit the notion of the conserved current and of the gravitational energy-momentum tensor has not an invariant sense. Although, this object is invariant under diffeomorphism transformations of the manifold, it is not invariant under local $SO(1,3)$ transformations of the coframe. Thus the conserved current of $\mathrm{GR}_{||}$ inherited from the general free parametric model is

no more than a type of a pseudo-tensor. Its connection to the Møller pseudo-tensor is shown in [18].

5. Conclusions and discussion

We considered a general 3-parametric teleparallel model in a coframe representation. The field equations and the conserved current (vector-valued 3-form) are derived via the variation procedure. By using the Noether technique the current is shown to be produce by the invariance of the Lagrangian under diffeomorfism transformations of the coframe. Consequently it is the energy-momentum current. The energy momentum-tensor of the coframe field is constructed. We considered a minimal coupling system of a coframe and a material field. It is shown that the total energy-momentum current of the system plays a role of the source of the coframe field strength. The total current is conserved, which yields a possibility of redistributing the of the energy between the coframe and the material field. Such effect is forbidden in the framework of the standard GR. A special case of the teleparallel equivalent of GR is discussed. This model is derived from the general construction by a specification of the free parameters. However, the conserved current of this equivalent of GR has not an invariant sense. It is because of localization of the Lorentz symmetry.
The result is: The standard GR has in the parametric space a neighborhood of viable models with the same Schwarzschild solutions. This models however have a better Lagrangian behavior and produce an invariant energy-momentum tensor.
The study of general teleparallel models can be interesting from two points of view:
1) As a family of *viable alternative models* of gravity. For this the parameters should be taken as $\rho_1 = 0$ and $\lambda = \rho_2 + 2\rho_3 \neq 0$. Certainly this teleparallel construction is different from the Einstein theory in the treatment of the axial symmetric spaces. The exact solution of such type can give a good indication of viability of this alternative model. Another line is to study alternative models of coupling of gravity with material fields. It is also important to search for a bound on the parameter λ.
2) The coframe approach can serve as an *alternative formulation of the standard GR.* This formalism can be helpful for the treatment of the energy-momentum problem of GR in the integral (quasi-local) aspect. An appropriative defined integral of the non-local invariant teleparallel current hoped to preserve the local invariance. The examples of such type behavior are well known from the electromagnetism theory.

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Appendix A. Basic notations and definitions

Let us list our basic conventions. We consider an n-dimensional differential manifold M of signature

$$\eta_{ab} = diag(-1, +1, \cdots, +1). \tag{A.1}$$

Let the manifold M will be endowed with a smooth coframe field (1-forms)

$$\{\vartheta^a(x),\ a = 0, \cdots, n-1\}. \tag{A.2}$$

Note that a smooth non-degenerate frame (coframe) field can be defined on a manifold of a zero second Stiefel-Whitney class. However this topological restriction is not exactly relevant in physics because the solutions of physical field equations can degenerate at a point or on a curve. Moreover, these solutions produce the most important physical models (particles, strings, etc.).

The coframe $\vartheta^a(x)$ represents, at a given point $x \in M$, a basis of the linear space of 1-forms Ω^1. The set of all non-zero exterior products of basis 1-forms

$$\vartheta^{a_1,\cdots,a_p} := \vartheta^{a_1} \wedge \cdots \wedge \vartheta^{a_p} \tag{A.3}$$

represents a basis of the linear space of p-forms Ω^p. Note the (anti)commutative rule for arbitrary forms $\alpha \in \Omega^p$ and $\beta \in \Omega^q$

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha. \tag{A.4}$$

The dual set of vector fields

$$\{e_a(x),\ a = 0, \cdots, n-1\} \tag{A.5}$$

forms a basis of the linear space of vector fields at a given point.

The duality of vectors and 1-forms can be expressed by *inter product* operation for which we use the symbol $\rfloor$. Namely,

$$e_a \rfloor \vartheta^b = \delta^b_a. \tag{A.6}$$

The action $X \rfloor w$ of a vector X on a form w of arbitrary degree p is defined by requiring: (i) linearity in X and in w, (ii) modified Leibniz rule for the wedge product of $\alpha \in \Omega^p$ and $\beta \in \Omega^q$

$$X \rfloor (\alpha \wedge \beta) = (X \rfloor \alpha) \wedge \beta + (-1)^p \alpha \wedge (X \rfloor \beta). \tag{A.7}$$

These properties together with (A.6) guarantee the uniqueness of the map $\rfloor : \Omega^p \to \Omega^{p-1}$.

The following relations involving the inner product operation ($p = deg(w)$) are useful for actual calculations.

$$X \rfloor (Y \rfloor w) \quad = \ - Y \rfloor (X \rfloor w), \tag{A.8}$$

$$\vartheta^a \wedge (e_a \rfloor w) = pw, \tag{A.9}$$

$$e_a \rfloor (\vartheta^a \wedge w) = (n-p)w. \tag{A.10}$$

We use also the forms $\vartheta_a := \eta_{ab}\vartheta^b$ with subscript and the corresponding vector fields $e^a := \eta^{ab}e_b$ with superscript. Thus

$$e_a \rfloor \vartheta_b = \eta_{ab}. \tag{A.11}$$

The linear spaces Ω^p and Ω^{n-p} have the same dimensions $\binom{n}{k} = \binom{n}{n-p}$. Thus they are isomorphic. This isomorphism *Hodge dual map* is linear. Thus it is enough to define its action on basis forms:

$$*\left(\vartheta^{a_1\cdots a_p}\right) = \frac{1}{(n-p)!}\epsilon^{a_1\cdots a_p a_{p+1}\cdots a_n}\vartheta_{a_{p+1}\cdots a_n}. \tag{A.12}$$

We use here the complete antisymmetric pseudo-tensor $\epsilon^{a_1\cdots a_{n-1}}$ which is normalized as $\epsilon^{01\cdots(n-1)} = 1$. The set of indices $\{a_1,\cdots,a_n\}$ is an even permutation of the standard set $\{0,1,\cdots,(n-1)\}$.
Thus $*\vartheta^{0\cdots(n-1)} = 1$ and $*1 = -\vartheta^{0\cdots(n-1)}$.
The consequence of the definition (A.12) is ($deg(\alpha) = deg(\beta)$)

$$\alpha \wedge *\beta = \beta \wedge \alpha. \tag{A.13}$$

For the choice of the signature (A.1) we obtain

$$*^2\, w = (-1)^{p(n-p)+1}w. \tag{A.14}$$

In the case $n = 4$ the operator $*^2$ preserves the forms of odd degree and changes the sign of the forms of even degree.
The following equation is useful for actual calculations

$$e_a \rfloor w = - * (\vartheta_a \wedge *w). \tag{A.15}$$

To prove this linear relation it is enough to check it for the basis forms.
The pseudo-orthonormality for the basis forms ϑ^a yields the *metric tensor* g on the manifold M

$$g = \eta_{ab}\vartheta^a \otimes \vartheta^b. \tag{A.16}$$

The formulas (A.11) and (A.15) can be applied to derive a useful form of a scalar product of two vectors X and Y. We write these vectors in the basis e_a as $X = X^m e_m$ and $Y = Y^m e_m$. Thus the scalar product is

$$< X, Y >= X^m Y^n < e_m, e_n >= X^m Y^n \eta_{mn}.$$

Using (A.11) we obtain

$$< X, Y >= X^m Y^n (e_m \rfloor \vartheta_n)$$

Thus

$$< X, Y >= X \rfloor {}^\sharp Y = Y \rfloor {}^\sharp X, \tag{A.17}$$

where ${}^\sharp X$ is the 1-form dual to the vector X which obtained by a canonical map from vectors to 1-forms

$$\sharp : X^m e_m \to X^m \vartheta_m.$$

Appendix B. Equivalence of (2.6) and (3.24)

Two forms of the field equation are linear in the coefficients ρ_i. Thus it is enough to prove the equivalence for separately for every parameter.
Start with the ρ_1-terms by taking $\rho_2 = \rho_3 = 0$ in both equations. The conjugated momentum (3.10) in this case

$$\mathcal{F}_a = \rho_1 d\vartheta_a = \rho_1 \mathcal{C}_a \tag{B.1}$$

Insert this expression in the LHS of the equation (2.6)

$$\begin{aligned}
&\rho_1\Big(2d * d\vartheta_a + e_a\rfloor(d\vartheta^b \wedge *d\vartheta_b) - 2(e_a\rfloor d\vartheta^b) \wedge *d\vartheta_b\Big)\\
&\qquad = 2\Big[d * \mathcal{F}_a - \Big((e_a\rfloor \mathcal{C}^b) \wedge *\mathcal{F}_b + \frac{1}{2}e_a\rfloor(\mathcal{C}^b \wedge *\mathcal{F}_b)\Big)\Big]\\
&\qquad\quad = 2(d * \mathcal{F}_a - \mathcal{T}_a)
\end{aligned} \tag{B.2}$$

Consider the ρ_2-terms by taking $\rho_1 = \rho_3 = 0$. The congugated momentum (3.10) in this case

$$\mathcal{F}_a = \rho_2 e_a\rfloor(d\vartheta^m \wedge \vartheta_m) \tag{B.3}$$

Consequently by using (A.15)

$$*\,\mathcal{F}_a = \rho_2\Big(\vartheta_a \wedge *(d\vartheta^m \wedge \vartheta_m)\Big) \tag{B.4}$$

and

$$d * \mathcal{F}_a = \rho_2\Big(d\vartheta_a \wedge *(d\vartheta^m \wedge \vartheta_m) - \vartheta_a \wedge d * (d\vartheta^m \wedge \vartheta_m)\Big) \tag{B.5}$$

Insert the expressions (B.4) and (B.5) into the ρ_2-term of the LHS of the equation (2.6) to obtain

$$\begin{aligned}
&\rho_2\Big(- 2\vartheta_a \wedge d * (d\vartheta^b \wedge \vartheta_b) + 2d\vartheta_a \wedge *(d\vartheta^b \wedge \vartheta_b) +\\
&\qquad e_a\rfloor\Big(d\vartheta^c \wedge \vartheta_c \wedge *(d\vartheta^b \wedge \vartheta_b)\Big) - 2(e_a\rfloor d\vartheta^b) \wedge \vartheta_b \wedge *(d\vartheta^c \wedge \vartheta_c)\Big) +\\
&\qquad = 2\Big[d * \mathcal{F}_a - \Big((e_a\rfloor \mathcal{C}^b) \wedge *\mathcal{F}_b + \frac{1}{2}e_a\rfloor(\mathcal{C}^b \wedge *\mathcal{F}_b)\Big)\Big]\\
&\qquad\quad = 2(d * \mathcal{F}_a - \mathcal{T}_a)
\end{aligned}$$

As for the ρ_3-terms we take $\rho_1 = \rho_2 = 0$. The conjugated momentum (3.10) in this case

$$\mathcal{F}_a = \rho_3\Big(d\vartheta_a - \vartheta_a \wedge (e_m\rfloor d\vartheta^m)\Big) = \rho_3 e_m\rfloor(d\vartheta^m \wedge \vartheta_a) \tag{B.6}$$

Consequently by using (A.15)

$$*\,\mathcal{F}_a = \rho_3\vartheta_m \wedge *(d\vartheta^m \wedge \vartheta_a)\Big) \tag{B.7}$$

and

$$d * \mathcal{F}_a = \rho_3\Big(d\vartheta_m \wedge *(d\vartheta^m \wedge \vartheta_a) - \vartheta_m \wedge d * (d\vartheta^m \wedge \vartheta_a)\Big) \tag{B.8}$$

Insert the expressions (B.7) and (B.8) into the ρ_2-term of the LHS of the equation (2.6) to obtain

$$\begin{aligned}
\rho_3\Big(& -2\vartheta_b\wedge d*(\vartheta_a\wedge d\vartheta^b)+2d\vartheta_b\wedge *(\vartheta_a\wedge d\vartheta^b)+\\
& e_a\rfloor\Big(\vartheta_c\wedge d\vartheta^b\wedge *(d\vartheta^c\wedge\vartheta_b)\Big)-2(e_a\rfloor d\vartheta^b)\wedge\vartheta_c\wedge *(d\vartheta^c\wedge\vartheta_b)\Big)\\
& =2\Big[d*\mathcal{F}_a-\Big((e_a\rfloor\mathcal{C}^b)\wedge *\mathcal{F}_b+\frac{1}{2}e_a\rfloor(\mathcal{C}^b\wedge *\mathcal{F}_b)\Big)\Big]\\
& =2(d*\mathcal{F}_a-\mathcal{T}_a)
\end{aligned}$$

Consequently the equivalence of two forms of the field equation is proven for all values of the parameters.

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